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SEMANTICS OF INFINITE TREE LOGIC PROGRAMMING*

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Abstract. We address the problem of declarative and operational semantics for logic programming in the domain of infinite trees. We consider logic programming semantics based on the now familiar function T_P which maps from and into interpretations of the program P. The main point of departure of our work from the literature is that we include unequations in our treatment. Specifically, we prove that the intuitive notions of success and finite failure, defined in terms of T_P, exactly correspond to the operational semantics. The corresponding proofs in the case where no unequations are considered are relatively straightforward mainly because the function T_P has a closure property with respect to a suitable metric space of infinite trees. When unequations are considered, however, the function loses this property and consequently the proofs become more complex. The key to our treatment is a result about images of T_P; we show that these sets have a property analogous to closure. Finally, we also prove certain results pertaining to infinite derivations. These concern the greatest fixpoint of T_P and the concept of completed logic programs and negation-as-failure.

. Introduction

In this paper, we address the problem of declarative and operational semantics ɔr logic programming in the domain of infinite trees. This programming concept 'as pioneered by Colmerauer in his programming language PROLOG II [2-4]. Infinite ·ee logic programs, in our treatment, have two novel differences from traditional ɔgic programs: first, they accommodate the assertion of equality/unequality between ːrms in their clauses, and second, they use a different unification algorithm in btaining derivation sequences.

The effort herein is motivated by two main reasons:

(1) While PROLOG II has an accompanying theoretical model [2], this model is rimarily based upon operational concepts. Logic programs are considered to be ːrm rewriting systems whose basic derivation step is that of reducing one system f equations and unequations into another. Algebraic considerations, with respect ı the algebra of infinite trees, are restricted to those processes in PROLOG II which eal with the solving of systems of equations and unequations.

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(2) Most other efforts dealing with aspects of the formal semantics of infinite tree logic programming completely omit unequations in the programs. For example, Van Emden and Lloyd [8] prove that PROLOG II, without unequations, is sound with respect to a certain equality theory. A complete logical basis to PROLOG II is given by Jaffar et al. [9] wherein they show that, with respect to another theory, PROLOG II, without unequations, is sound and complete for both successful and finitely failed derivations, and also that PROLOG II is sound and complete for negation-as-failure. A more algebraic approach is taken in [1] which establishes some fundamental topological properties of the infinite trees associated with the semantics of logic programs. See also [13, Chapter 4] for a discussion along these lines.

We consider logic programming semantics based on the now familiar function T_P associated with a program P. Such functions map from and into interpretations. In our case, these interpretations are defined like Herbrand interpretations except that we consider terms to be potentially infinite in size. We prove that the intuitive notions of success and finite failure, defined in terms of T_P correspond exactly to the operational semantics of PROLOG II. The corresponding proofs for the case where no unequations are considered is relatively straightforward mainly because the function T_P has a closure property with respect to a suitable metric space of infinite trees (see, e.g., [13, Chapter 4]). When unequations are considered, however, the function loses this closure property and consequently the proofs become more complex. We give an example which illustrates this in the next section.

In summary, the primary results of this paper deal with those derivations which are either successful or finitely failed—in either case, they are finite derivations. We prove that they correspond to the intuitive notions of success and finite failure. The key to our proofs is a result about images of T_P; we show that these sets have *rational covers*, a property analogous to that enjoyed by closed sets. Additionally, we prove certain results pertaining to infinite derivations. These concern the greatest fixpoint of T_P and the concept of completed logic programs and negation-as-failure.

2. Preliminaries

We use the symbols Σ, Π and V to denote our denumerable collections of functors, predicate symbols and variables respectively. $\tau(\Sigma)$ and $\tau(\Sigma \cup V)$ denote the (ground) finite trees and the finite terms respectively. We extend this notation to (ground) infinite trees and infinite terms with $\tau^*(\Sigma)$ and $\tau^*(\Sigma \cup V)$ respectively. Subsets of these, $\tau^{\mathrm{R}}(\Sigma)$ and $\tau^{\mathrm{R}}(\Sigma \cup V)$ denoting the *rational trees* and *rational terms* respectively, contain those trees and terms which have a finite number of subtrees and subterms respectively (see e.g. [6]). Throughout this paper, we assume that Σ contains at least one functor of arity 0 and one functor of arity $\geqslant 1$. Thus $\tau(\Sigma)$ is infinite. Finally, we define the *infinitary Herbrand Base* HB* to be

$$\{p(t_1, \ldots, t_n) : p \text{ is an } n\text{-ary symbol in } \Pi \text{ and } t_i \in \tau^*(\Sigma), 1 \leqslant i \leqslant n\}.$$

ın *atom* is of the form $p(t_1, \ldots, t_n)$ where p is an n-ary symbol in Π and $t_i \in$ $^*(\Sigma \cup V)$, $1 \leq i \leq n$. A *finite atom* is defined like this except we use τ in place of *. Similarly for *rational atoms. Equations and unequations* are of the form $s = t$ nd $s \neq t$, where s and t are finite terms, respectively. A (finite or infinite) *system* ; then defined to be a set of equations and unequations.

A (ground) *substitution* α is an idempotent mapping from the set of variables ı V into $\tau^*(\Sigma \cup V)$ $(\tau^*(\Sigma))$. Where the image of such α falls into $\tau^{\mathrm{R}}(\Sigma \cup V)$, we all α a *rational substitution.* As usual, we can apply substitutions to other kinds of bjects such as substitutions, equations, unequations, atoms, etc. For example, we ıay say that α is an instance of β, denoted $\alpha \leq \beta$, if there exists γ such that $\beta\gamma = \alpha$.

We say that an equation $(s = t)\alpha$ is *true* if $s\alpha$ and $t\alpha$ are identical. We denote ıis by $s\alpha = t\alpha$. Similarly, we say that an unequation $(s \neq t)\alpha$ is *true* if $s\alpha$ and $t\alpha$ re not identical. We denote this by $s\alpha \neq t\alpha$. Where S is a system, $S\alpha$ is true if $e\alpha$ nd $\bar{e}\alpha$ are true for all equations e and unequations $\bar{e}$ in S. It is very important to ote that $S\alpha$ being true does not imply that $S\alpha\beta$ is true for all β. (Take for example $= \{x \neq f(a)\}$, $\alpha(x) = f(y)$ and $\beta(y) = a$.)

A *ground satisfier* α of a system S is a ground substitution such that $S\alpha$ is true. *satisfier* β of a system S is a substitution such that $S\alpha$ is true for every ground ıstance α of β. Thus every instance of a satisfier of S is also a satisfier of S. A ystem S is *solvable* if S has a ground satisfier. We define *most general unifiers* ngu's) α of systems of equations (and not unequations) E in the usual way, i.e. is a satisfier of E such that $\alpha \geq \beta$ for all satisfiers β of E. $S1 \models S2$ means that very satisfier of $S1$ is also a satisfier of $S2$. *Equivalent* systems $S1$ and $S2$ then are ıch that $S1 \models S2$ and $S2 \models S1$.

A *clause* and a *goal* are respectively of the form

$$A \leftarrow (S \,\Box\, B_1, B_2, \ldots, B_n) \quad \text{and} \quad (S \,\Box\, B_1, B_2, \ldots, B_n),$$

here $n \geq 0$, A and B_i, $1 \leq i \leq n$, are finite atoms and S is a finite system. As usual, e call A the *head* of the clause. A *program* is a finite collection of clauses.

We define an *interpretation* I to be a subset of HB*. We write I_p, where $p \in \Pi$, to e the restriction of I to those elements involving p. I is a *model* of a program P , for each clause

$$p(\tilde{x}) \leftarrow (S \,\Box\, B_1, B_2, \ldots, B_n)$$

ı P and each ground satisfier α of S, $p(\tilde{x})\alpha \in I_p$ whenever $B_i\alpha \in I$ for all $1 \leq i \leq n$. /e will also write $I \models P$ when I models P. Clearly we can extend interpretations I ı apply to closed first-order formulas Q which use the alphabet $\Pi \cup \Sigma$; whenever ıch Q has a model I, we yet again use the notation $I \models Q$.

A *P-derivation* of G, where P and G are a program and goal respectively, is a inite or infinite) non-empty sequence of goals $G = G_0, G_1, G_2, \ldots$ such that (a) ıch G_i in the sequence is of the form $(S \,\Box\, B_1, \ldots, B_n)$ where S is solvable and ≥ 0. If $n = 0$, G_i is the last goal in the derivation. Otherwise, (b) there exists a collection of n variants of clauses in P, say

$$A_1 \leftarrow (S_1 \Box \tilde{C}_1),$$
$$A_2 \leftarrow (S_2 \Box \tilde{C}_2),$$
$$\vdots$$
$$A_n \leftarrow (S_n \Box \tilde{C}_n)$$

such that

$$S' = S \cup S_1 \cup S_2 \cup \cdots \cup S_n \cup \{B_1 = A_1, B_2 = A_2, \ldots, B_n = A_n\}$$

is solvable and (c) G_{i+1} is $(S' \Box \tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_n)$. Note that in P-derivations, a system in a goal contains all systems in preceding goals.

A P-derivation is *successful* if its last goal contains no atoms, i.e. the last goal can be written in the form $(S \Box \emptyset)$ where $\emptyset$ denotes the empty sequence of atoms. A P-derivation is *finitely failed* if it is not successful and its length is finite. All other P-derivations are *infinite.* The *derived system* of a finite P-derivation is the system in its last goal; in the case of infinite derivations, the derived system is the union of all the systems appearing therein.

The *success set* $SS(P)$ of a program P is the set of goals $G = (S \Box A)$ where there exists a successful P-derivation of G whose derived system S' is such that every satisfier of S extends to a satisfier of S'. This intuitively means that S' is equivalent to S except for new variables in S'. The *finite failure set* $FF(P)$ of a program P is the set of goals $G = (S \Box A)$ all of whose P-derivations are finitely failed. Let $SS_n(P)$, $n \geq 0$, be the subset of $SS(P)$ wherein those goals have one P-derivation of length $\leq n$; $FF_n(P)$, on the other hand, is the subset of $FF(P)$ wherein the goals have no P-derivation longer than n.

The set of ground instances of an atom A is denoted by $[A]$. We can extend this notation to goals containing one atom: $[(S \Box B)] = \{B\alpha : \alpha \text{ is a ground satisfier of } S\}$.

We now present a number of fundamental results required by the following sections in this paper. The first two of these concern the solvability of systems:

Lemma 2.1. (a) *A solvable finite system of equations E has a rational mgu.*

(b) *For every rational term $\tilde{t}$, there exists a finite system of equations containing distinguished variables $\tilde{x}$ and whose mgu α is such that $\tilde{x}\alpha = \tilde{t}$.*

Proof. See, e.g., [6]. $\Box$

Lemma 2.2. *A possibly infinite system S containing equations E and a finite number of unequations $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$, $n \geq 1$, is solvable iff each subsystem $E \cup \{\bar{e}_i\}$ is separately solvable, $1 \leq i \leq n$.*

Proof. [5]. $\Box$

The next lemma concerns successful and finitely failed P-derivations. Its proof ; straightforward.

emma 2.3. (a) *If the goal* $(S \,\Box\, A_1, A_2, \ldots, A_n)$ *has a successful P-derivation of* *ngth n, then* $(S \,\Box\, A_i)$, $1 \leq i \leq n$, *has a successful P-derivation of length* $\leq n$.

(b) *If the goals* $(S_i \,\Box\, A_i)$, $1 \leq i \leq n$, *each has a successful P-derivation and* $S = S_1 \cup \cdots \cup S_n$ *is solvable, then* $(S_1 \cup \cdots \cup S_n \,\Box\, A_1, \ldots, A_n)$ *has a successful P-derivation.*

(c) *If the goal* $(S \,\Box\, \tilde{A}) \in \mathrm{FF}_n(P)$, $n \geq 1$, *and the system* $S \cup S'$ *is solvable, then* $S \cup S' \,\Box\, \tilde{A}) \in \mathrm{FF}_n(P)$.

(d) *If the equations E in the finite system S have a ground mgu and if* $S \,\Box\, B_1, B_2, \ldots, B_n)$, $n \geq 1$, *have only finitely failed P-derivations of length* $\leq m$, *then* *iere exists an i,* $1 \leq i \leq n$, *such that* $(S \,\Box\, B_i) \in \mathrm{FF}_m(P)$.

The main issue of this paper, the semantics of the class of infinite tree logic rograms defined above, involves the central concept of the 'immediate consequence' inction T_P of a program P. Similar to the corresponding functions in the literature, g. [7, 10], our function maps from and into 2^{HB^*} and is defined as

$$
\begin{aligned}
T_P(I) = \{A : {} & \text{there is a ground substitution } \alpha \text{ and a clause in } P\\
& B \leftarrow (S \,\Box\, C_1, \ldots, C_n),\ n \geq 0, \text{ such that}\\
& \quad \text{(a) } B\alpha = A,\\
& \quad \text{(b) } \alpha \text{ is a ground satisfier of } S, \text{ and}\\
& \quad \text{(c) } C_i\alpha \in I,\ 1 \leq i \leq n\}.
\end{aligned}
$$

Consider the metric on infinite trees as given in [13, Chapter 4]. When P does ot involve unequations, T_P maps closed sets under this metric to closed sets. We ın now show with an example that, in general, T_P does not enjoy this closure roperty:

$$p(x) \leftarrow (x = a \,\Box\, q(y)), \qquad q(x) \leftarrow (x = f(y), x \neq f(x) \,\Box\, q(y)).$$

/hile HB* is closed, it is easy to see that $T_P(\mathrm{HB}^*)$ is not because the limit element $(f(f(f(\ldots)$ is not in $T_P(\mathrm{HB}^*)$.

The least and greatest fixpoints (with respect to the $\subseteq$-ordered set 2^{HB^*}) of such ınctions T_P can easily be shown to exist; let $\mathrm{lfp}(T_P)$ and $\mathrm{gfp}(T_P)$ respectively enote them. Following standard terminology, we define

$$T_P\!\uparrow\! 0 = \{\},$$

$$T_P\!\uparrow\! z = \begin{cases} T_P(T_P\!\uparrow\!(z-1)), & \text{if } z \text{ is a successor ordinal,}\\ \bigcup_{y<z} T_P\!\uparrow\! y, & \text{if } z \text{ is a limit ordinal,}\end{cases}$$

$$T_P\!\downarrow\! 0 = \mathrm{HB}^*,$$

$$T_P\!\downarrow\! z = \begin{cases} T_P(T_P\!\downarrow\!(z-1)), & \text{if } z \text{ is a successor ordinal,}\\ \bigcap_{y<z} T_P\!\downarrow\! y, & \text{if } z \text{ is a limit ordinal.}\end{cases}$$

Our final lemma deals with T_P. Once again we omit proofs because they are adaptable from corresponding proofs in, e.g. [13].

Lemma 2.4. (a) *T_P is continuous.*

(b) $T_P\uparrow\omega = \mathrm{lfp}(T_P) \subseteq T_P\downarrow z = \mathrm{gfp}(T_P) \subseteq T_P\downarrow\omega$ *for some ordinal* $z \geqslant \omega$.

We finish off this section now with some conventions we adopt for notational convenience and ease of proofs. Firstly, the letters E, $\bar{E}$, S, P, T, SS and FF shall, unless otherwise stated, stand for equations, unequations, systems, programs, the function T_P associated with P, SS(P) and FF(P) respectively. We use possibly subscripted symbols w, x, y and z to denote variables. We use possibly subscripted symbols A, B, C and D to denote atoms. We use possibly subscripted symbols p and q to denote predicate symbols. We use possibly subscripted symbols f and g to denote functors, and the symbols a and b to denote constant symbols. We use possibly subscripted Greek symbols to denote substitutions. In a P-derivation of a goal G, we use the notation $G = G_0, G_1, \ldots$ where the system S_i in G_i is denoted $E_i \cup \bar{E}_i$, $i \geqslant 0$. In case such a sequence is infinite, we write S_ω to denote the union of the S_i, $i \geqslant 0$.

Next, we use the symbol $\sim$ to denote finite sequences of objects such as terms, trees, atoms, clauses, etc. Thus, e.g. $\tilde{s} = \tilde{t}$ may denote the finite system of equations $\{s_1 = t_1, \ldots, s_n = t_n\}$. We sometimes, by a slight abuse of notation, will write $A = B$; this clearly means the finite system of equations $\tilde{s} = \tilde{t}$ where A and B are of the form $p(\tilde{s})$ and $p(\tilde{t})$.

Our final convention regards the syntax of programs: each clause is of the form

$$p(\tilde{x}) \leftarrow (E(\tilde{x}, \tilde{y}) \cup \bar{E}(\tilde{y}) \;\Box\; B_1, B_2, \ldots, B_n),$$

where

(a) $n \geqslant 0$,

(b) $\tilde{x}$ and $\tilde{y}$ have no common variable,

(c) the collection of variables in the atoms B_i, $1 \leqslant i \leqslant n$, are contained in $\tilde{y}$, and

(d) B_i and B_j, $i \neq j$, have no common variables.

We also specify that the clauses using the same predicate symbol in the head are written with identical heads. It is easy to see that any program $P1$ can be rewritten to an 'equivalent' form $P2$ allowable by the above restriction. The important point, however, is that all our results below apply equivalently to $P1$ and $P2$.

3. Lemmas on rationality

This section contains the backbone to the main results of this paper. The focus of attention here is certain images of the transformation function T. We prove that these sets have the following *rational cover* property: a set Q of (finite and infinite) ground atoms has a rational cover if for each $p(\tilde{t}) \in Q$, there exists a rational atom

$p(\tilde{w})$ such that $p(\tilde{t}) \in [p(\tilde{w})] \subseteq Q$. Intuitively, if Q has a rational cover, then Q contains no 'isolated' irrational atoms.

Lemma 3.1. *Every finite system S with ground satisfier α has a rational satisfier $\zeta \geq \alpha$.*

Proof. Let S be the union of equations E and unequations $\bar{E} = \{s_1 \neq t_1, s_2 \neq t_2, \ldots, s_n \neq t_n\}$, $n \geq 0$, and let $S\alpha$ be true. Let β_0 be the mgu of E (cf. Lemma 2.1); thus β_0 is rational. Note that there exists δ such that $\alpha = \beta_0\delta$ since β_0 is an mgu. Consider two possible cases:

(1) Either $n = 0$ or otherwise each of $s_i\beta_0 = t_i\beta_0$, $1 \leq i \leq n$, is unsolvable;

(2) There exists $1 \leq i \leq n$ such that $s_i\beta_0 = t_i\beta_0$ has a rational mgu γ.

In case (1), we are clearly finished with β_0 as the desired ζ.

Suppose then we have case (2). Without loss of generality, suppose $s_1\beta_0\gamma = t_1\beta_0\gamma$. Now, $s_1\beta_0\delta \neq t_1\beta_0\delta$ and hence γ differs from δ. We seek now a rational substitution β_1 such that $\delta \leq \beta_1$ and *not* $\beta_1 \leq \gamma$.

Firstly, let n be the depth of a node at which the two ground trees $s_1\beta_0\delta$ and $t_1\beta_0\delta$ differ. Now we define β_1 as follows: for every variable x mapped into, say v, by δ, β_1 maps x into v' where v' is any finite term which is identical to v up to depth n *and* $v' \geq v$. The important property that β_1 has is that the tree $s_1\beta_0\beta_1$ agrees with $s_1\beta_0\delta$ on every path down to a leaf or to a depth of $\geq n$. Similarly, $t_1\beta_0\beta_1$ agrees with $t_1\beta_0\delta$ on every path down to a leaf or to a depth of $\geq n$. Thus no instance of $s_1\beta_0\beta_1$ is equal to an instance of $t_1\beta_0\beta_1$. It is now easily verifiable that $\beta_0\beta_1 \geq \alpha$ and $\beta_0\beta_1$ is a rational satisfier of $E \cup \{s_1 \neq t_1\}$.

The proof is now complete by repeating the above argument if necessary; this time we use $\beta_0\beta_1$ in place of β_0 and $s_2 \neq t_2$ in place of $s_1 \neq t_1$. Clearly the number k of such repetitions cannot exceed n. The desired ζ is therefore $\beta_0\beta_1 \ldots \beta_k$. $\square$

Lemma 3.2. *For all finite n, $T\uparrow n$ has a rational cover.*

Proof. We use induction on n. The lemma is trivially true when $n = 0$. For the induction step, let $p(\tilde{t}) \in T\uparrow n$, $n \geq 1$. Thus there exists a clause

$$p(\tilde{x}) \leftarrow (E \cup \bar{E} \,\square\, B_1, B_2, \ldots, B_m) \tag{3.1}$$

where $m \geq 0$, such that for some ground substitution α, (a) $p(\tilde{x})\alpha = p(\tilde{t})$, (b) $(E \cup \bar{E})\alpha$ is true. (c) $B_i\alpha \in T\uparrow(n-1)$, for all $1 \leq i \leq m$, and using the induction hypothesis on (c), there exist rational atoms B_i' such that

$$B_i\alpha \in [B_i'] \subseteq T\uparrow(n-1)$$

for all $1 \leq i \leq m$. Now we can construct B_i' such that no variable in B_i' appears in B_j', for all $1 \leq i \neq j \leq m$. Thus

$$E \cup \{B_1 = B_1', B_2 = B_2', \ldots, B_m = B_m'\} \cup \bar{E}$$

is solvable. This is because of the following: (b) above says that α solves $(E \cup \bar{E})$. Thus for all $1 \leq i \leq m$, $B_i\alpha = B_i'\beta_i$ where the $m+1$ substitutions $\alpha, \beta_1, \ldots, \beta_m$ do not have variables in common. Clearly $\alpha\beta_1 \ldots \beta_m$, an extension of α, is a satisfier of the system.

The final proof step is provided by Lemma 3.1, i.e. there must exist a rational satisfier $\gamma \geq \alpha\beta_1 \ldots \beta_m$ of the system. Using the fact that $[B_i\gamma] = [B_i'\gamma] \subseteq [B_i']$ for $1 \leq i \leq m$, it is a simple matter to check with the definition of T that all instances of the desired rational atom $p(\tilde{x})\gamma$ are in $T\uparrow n$. $\square$

Lemma 3.3. *For all finite n, $T\downarrow n$ has a rational cover.*

Proof. Using induction on n, the lemma is trivially true when $n = 0$. (Take the term x.) For the induction step, the proof is almost identical to the above and is, therefore, omitted. $\square$

We now consider the complements of the sets considered above. The proofs below are slightly more complicated and require a finite presentability property of the sets $T\uparrow n$ and $T\downarrow n$, $n \geq 0$. Let $U \subseteq \tau^*(\Sigma)^n$, $n \geq 1$; that is, U is a possibly infinite set of n-tuples of infinite trees. We say that U is *finitely presentable* if there is a finite collection of finite systems $\{S_1, S_2, \ldots, S_m\}$, $m \geq 0$, containing n distinguished variables $\tilde{x}$ such that $\tilde{t} \in U$ iff there exists a ground satisfier α of S_i, for some $1 \leq i \leq m$, such that $\tilde{x}\alpha = \tilde{t}$.

We can extend this notion to interpretations I_q restricted to one predicate symbol q of arity $n \geq 1$. That is, I_q is finitely presentable if there is a finite collection of finite systems $\{S_1, S_2, \ldots, S_m\}$, $m \geq 0$, containing n distinguished variables $\tilde{x}$ such that $q(\tilde{t}) \in I_q$ iff there exists a ground satisfier α of S_i, for some $1 \leq i \leq m$, such that $\tilde{x}\alpha = t$. Finally, an interpretation I is finitely presentable if the subsets I_q of I corresponding to distinct predicate symbols q in I are each finitely presentable.

Lemma 3.4. (a) *$T\uparrow n$ is finitely presentable, $n \geq 0$.*
(b) *$T\downarrow n$ is finitely presentable, $n \geq 0$.*

Proof. We do (a) only since the proof of (b) is similar. Proceeding by induction on n, we first observe that (a) trivially holds in the base case $n = 0$. (For m-ary p, choose any inconsistent system containing m variables.)

The induction step is performed below over each m-ary predicate sysmbol q, i.e. we prove for all q that $(T\uparrow n)_q$, $n \geq 1$, is finitely presentable. Consider the collection C_q of all clauses in P of the form

$$q(\tilde{x}) \leftarrow (S \,\square\, q_1(\tilde{y}_1), q_2(\tilde{y}_2), \ldots, q_k(\tilde{y}_k))$$

where $k \geq 0$. Let I_i, $1 \leq i \leq k$, denote $(T\uparrow n-1)_{q_i}$. Using the induction hypothesis, let J_i be a finite presentation of I_i, $1 \leq i \leq k$. Without losing generality, assume that J_i has no variables in common with J_j, $i \neq j$, and each system S in J_i has $\tilde{y}_i$ as its distinguished variables. The essential step in this proof is given in the following construction: define J to contain solvable finite systems of the form

$$S \cup S_1 \cup S_2 \cup \cdots \cup S_k$$

where $S_i \in J_i$, $1 \leqslant i \leqslant k$. Thus J contains a finite number of systems. We finish the construction by obtaining the set $\tilde{J}$, the union of the sets J taken over all the clauses in C_q.

We now verify that $\tilde{J}$, with distinguished variables $\tilde{x}$, is indeed a finite presentation of $(T\uparrow n)_q$. Suppose $q(\tilde{t}) \in T\uparrow n$. There must then exist a clause, say of the above form, and a ground satisfier α of S such that for all $1 \leqslant i \leqslant k$, $q_i(\tilde{y}_i)\alpha \in T\uparrow(n-1)$. Using the induction hypothesis, each such $q_i\alpha$ is given by a ground satisfier of some system S_i above. Making these choices of S_i, $1 \leqslant i \leqslant k$, we have shown that $q(\tilde{t})$ is given by the ground satisfier of a system J in $\tilde{J}$. The remainder of the proof is that every ground satisfier α of a system J in $\tilde{J}$ gives $q(\tilde{t})\alpha \in T\uparrow n$. It is straightforward from the definition of T and the finite presentations S_i of the q_i, $1 \leqslant i \leqslant k$. □

In the proof below we sometimes abuse notation by writing rational terms in finite systems. We can safely do this by virtue of Lemma 2.1(b).

Lemma 3.5. *For all finite n,* $\overline{T\uparrow n}$ *has a rational cover.*

Proof. We proceed by a direct proof. Let $p(\tilde{t}) \in \overline{T\uparrow n}$, $n \geqslant 0$. We may as well assume that $p(\tilde{t})$ is irrational. Thus for every clause in P of the form

$$p(\tilde{x}) \leftarrow (E \cup \bar{E} \;\square\; B_1, B_2, \ldots, B_m) \tag{3.2}$$

every ground substitution α is such that

(a) $p(\tilde{x})\alpha \neq p(\tilde{t})$, or

(b) α is not a ground satisfier of $E \cup \bar{E}$, or

(c) $B_i\alpha \notin T\uparrow(n-1)$ for some $1 \leqslant i \leqslant m$.

In what follows, we will prove, for each such clause C of the form (3.2), that if $p(\tilde{t}) \in \overline{T\uparrow n}$, then there exists a rational atom $p(\tilde{t}') \geqslant p(\tilde{t})$ such that

$$p(\tilde{t}) \in [p(\tilde{t}')] \subseteq \overline{T_C(T\uparrow n-1)}.$$

The rest of the proof will then be presented using two facts:

(1) there exists a rational atom which is an instance of each of the rational atoms $p(\tilde{t}')$ obtained by considering these clauses C separately, and

(2) $(\overline{T\uparrow n})_p$ is equal to the intersection of the $\overline{T_C(T\uparrow n-1)}$, where C ranges over all the clauses of the form (3.2) in P.

Firstly, consider only those ground substitutions α where $p(\tilde{x})\alpha = p(\tilde{t})$; these α are the extensions of the unique substitution on $\tilde{x}$ which identifies $p(\tilde{x})$ and $p(\tilde{t})$. Thus for each such α, one of the conditions (b) or (c) above hold. Another way of putting this is as follows. Let J_i be a finite presentation of $(T\uparrow n-1)_{p_i}$ where p_i is the predicate symbol in B_i, $1 \leqslant i \leqslant m$. Let $F_i \in J_i$ for $1 \leqslant i \leqslant m$. Without losing generality, we assume that these F_i do not contain any unequations (since, for the purposes of this argument, we can include any unequations in $\bar{E}$ below). Now, every system of the form

$$E(\tilde{x}, \tilde{y}) \cup \bar{E}(\tilde{y}) \cup F_1 \cup F_2 \cup \cdots \cup F_m \tag{3.3}$$

does not admit any ground satisfier which is an extension of α (for otherwise $p(\tilde{x})\alpha \in T\uparrow n$). For each choice of a system (3.3) corresponding to the clause C (3.2), we will now find a rational atom $p(\tilde{t}'') \geq p(\tilde{t})$ such that the system

$$\{\tilde{x} = \tilde{t}''\} \cup (3.3)$$

is unsolvable. Since there are only a finite number of choices (3.3), there must exist a rational atom $p(\tilde{t}') \leq p(\tilde{t}'')$, for all $p(\tilde{t}'')$ corresponding to (3.3), such that $p(\tilde{t}) \in [p(\tilde{t}')]$. Essentially, $p(\tilde{t}')$ is the maximal common instance of the $p(\tilde{t}'')$. However, the important point to note is that $[p(\tilde{t}')] \subseteq T_C(T\uparrow n-1)$.

We now find $p(\tilde{t}'')$ corresponding to a clause C and a choice of a system (3.3). Let $E_t(\tilde{x})$ be a (infinite) system of equations whose only ground satisfier is such that $\tilde{x} = \tilde{t}$. We choose $E_t(\tilde{x})$ such that $\tilde{x}$ are its only variables in common with the system (3.3). Thus,

$$E_t(\tilde{x}) \cup E(\tilde{x}, \tilde{y}) \cup \bar{E}(\tilde{y}) \cup F_1 \cup F_2 \cup \cdots \cup F_m \tag{3.4}$$

is unsolvable.

Consider first the case where the subsystem

$$E(\tilde{x}, \tilde{y}) \cup F_1 \cup F_2 \cup \cdots \cup F_m \tag{3.5}$$

is unsolvable. Then, clearly, $p(\tilde{t}'') = p(\tilde{x})$ and we are done.

If, on the other hand, this subsystem (3.5) is solvable but

$$E_t(\tilde{x}) \cup E(\tilde{x}, \tilde{y}) \cup F_1 \cup F_2 \cup \cdots \cup F_m \tag{3.6}$$

is not, then consider $\tilde{x}\beta$ where β is an mgu of (3.5). It must be the case that $\tilde{t}$ is not an instance of $\tilde{x}\beta$. We now show that there is a number n such that all instances of $\tilde{x}\beta$ and $\tilde{t}$ differ at a node of depth n or less. We consider two cases:

(a) There is a node in $\tilde{t}$ which is different from the corresponding node in $\tilde{x}\beta$. Clearly n is the depth of this node.

(b) The only other case is when there are two instances of a variable z in $\tilde{x}\beta$ such that the two subtrees in corresponding positions in $\tilde{t}$ differ. Let m be the depth at which these two subtrees differ. If k is the maximum depth of the two nodes labeled with z in $\tilde{x}\beta$, then $n = m + k$.

In either case (a) or (b), the desired $p(\tilde{t}'')$ can be given by a finite atom such that $p(\tilde{t}'') \geq p(\tilde{t})$ and $p(\tilde{t}'')$ is identical to $p(\tilde{t})$ on every path down to a leaf node or to a depth of $\geq n$. For example, let $\tilde{x}$ and $\tilde{t}$ be a single variable and term, and see Fig. 1, then $p(\tilde{t}'')$ can be $f(g(a), f(g(g(w_1)), f(g(w_2), w_3)))$.

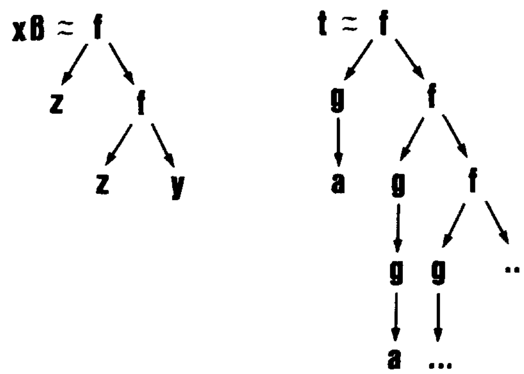

Fig. 1.

The only remaining possibility is that (3.6) is solvable. Here we apply Lemma 2.2 to have that

$$E_t(\tilde{x}) \cup E(\tilde{x}, \tilde{y}) \cup (s \neq u) \cup F_1 \cup F_2 \cup \cdots \cup F_m \tag{3.7}$$

is unsolvable, where $(s \neq u)$ is some unequation in $\bar{E}(\tilde{y})$. Consider $\tilde{x}\beta$, where β is an mgu of (3.5), and let $\tilde{z}$ denote the variables appearing in $\tilde{x}\beta$. If none of these variables $\tilde{z}$ appear in $s\beta$ or $u\beta$, then

$$E(\tilde{x}, \tilde{y}) \cup (s \neq u) \cup F_1 \cup F_2 \cup \cdots \cup F_m \tag{3.8}$$

is unsolvable. We argue this as follows. Note that $E_t(\tilde{x})$ and (3.5) have only $\tilde{x}$ as common variables. Because the only ground satisfier of $E_t(\tilde{x})$ is such that $\tilde{x} = \tilde{t}$, there exists a satisfier $\beta\gamma$ of (3.6) such that γ only maps variables in $E_t(\tilde{x})$ and $\tilde{z}$. Intuitively, γ equates $\tilde{x}\beta$ and $\tilde{t}$. Since neither of $s\beta$ and $u\beta$ contain variables appearing in $E_t(\tilde{x})$, $s\beta\gamma = s\beta$ and $u\beta\gamma = u\beta$. Because (3.7) is unsolvable, $s\beta\gamma = s\beta = u\beta = u\beta\gamma$. Hence (3.8) is unsolvable. We thus choose $p(\tilde{t}'') = p(\tilde{x})$ and we are done.

For the final case, we assume that $s\beta$ or $u\beta$ contains a variable in $\tilde{z}$. We use the hypothesis that (3.6) is solvable to obtain a substitution γ such that $\tilde{x}\beta\gamma = \tilde{t}$. As above, we have that γ maps only those variables appearing in $E_t(\tilde{x})$ and $\tilde{z}$. Now $s\beta\gamma = u\beta\gamma$ since (3.7) is unsolvable. Hence $\{s\beta = u\beta\}$ is solvable and has an mgu, say δ. Now δ only maps variables $\tilde{z}$ because $\delta \geq \gamma$. We now claim that we can choose $p(\tilde{t}'') = p(\tilde{x})\beta\delta$. The reason follows.

Any satisfier ζ of $\{\tilde{x} = \tilde{x}\beta\delta\} \cup (3.5)$ is an instance of $\beta\delta$ because δ only maps the variables $\tilde{z}$ in $\tilde{x}\beta$. This ensures that $\{s \neq u\}\zeta$ is false and hence $\{\tilde{x} = \tilde{x}\beta\delta\} \cup (3.3)$ is unsolvable. Intuitively this rational atom $p(t'') = p(\tilde{x})\beta\delta$ contains all the information required to enforce $s = u$.

In each of the three cases above, we have obtained, for each choice of $F_i \in J_i$, $1 \leq i \leq m$, a rational atom $p(\tilde{t}'')$ such that $p(\tilde{t}) \in [p(\tilde{t}'')]$ and no instance of $p(\tilde{t}'')$ is given by a satisfier of the system (3.3). Now there are a finite number of such $p(\tilde{t}'')$ and these $p(\tilde{t}'')$ share a common instance, namely $p(\tilde{t})$. Using the Lemmas 2.1 and 3.1, we have that there exists a rational atom $p(\tilde{t}') \leq p(\tilde{t}'')$, for all these $p(\tilde{t}'')$. Furthermore, $p(\tilde{t}) \in [p(\tilde{t}')] \subseteq T_C(T\uparrow n-1)$.

The last step of the proof is similar to the step above: the collection of rational atoms $p(\tilde{t}')$, each of which was obtained with respect to a particular clause C, has a common instance, namely $p(\tilde{t})$. Now there are a finite number of clauses C and the corresponding collection of $p(\tilde{t}')$ share a common instance, namely $p(\tilde{t})$. Thus there exists a rational atom $p(\tilde{w}) \leqslant p(\tilde{t}')$ for all $p(\tilde{t}')$ in this collection. Since the intersection of $\overline{T_C(T\uparrow n-1)}$, over all these clauses C, equals $\overline{T\uparrow n}$, $p(\tilde{t}) \in [p(\tilde{w})] \subseteq \overline{T\uparrow n}$. □

Lemma 3.6. *For all finite n, $\overline{T\downarrow n}$ has a rational cover.*

Proof. This proof is similar to the one above. □

4. Successful derivations

This section is entirely devoted to showing that the intuitive set of computed atoms, $T_P\uparrow\omega$, is given by the set of atoms actually computed, SS(P). As in most treatments of successful derivations in logic programming, the proofs below are relatively straightforward. The only possible delicacy in the proofs concerns the space of ground satisfiers of systems; these matters are accommodated easily by Lemma 3.1.

Theorem 4.1. $T_P\uparrow\omega = [\mathrm{SS}(P)]$.

Proof. We prove by induction that, for any goal $(E \cup \bar{E} \,\square\, p(\tilde{x})) \in \mathrm{SS}_n$, $[(E \cup \bar{E} \,\square\, p(\tilde{x}))] \subseteq T\uparrow n$, $n \geqslant 0$. The base case $n = 0$ is trivial. For the induction step, let $(E \cup \bar{E} \,\square\, p(\tilde{x})) \in \mathrm{SS}_n$, $n \geqslant 1$. Thus there exists a clause

$$p(\tilde{x}) \leftarrow (F \cup \bar{F} \,\square\, B_1, B_2, \ldots, B_m)$$

in P such that

$$(E \cup F \cup \bar{E} \cup \bar{F} \,\square\, B_1, B_2, \ldots, B_m)$$

has a successful P-derivation of length $\leqslant n-1$. By Lemma 2.3(a), $(E \cup F \cup \bar{E} \cup \bar{F} \,\square\, B_i)$, for all $1 \leqslant i \leqslant m$, has a successful P-derivation of length $\leqslant n-1$. Since $(E \cup \bar{E} \,\square\, p(\tilde{x})) \in \mathrm{SS}_n$, $(E \cup F \cup \bar{E} \cup \bar{F} \,\square\, B_i) \in \mathrm{SS}_{n-1}$, for all $1 \leqslant i \leqslant m$. By the induction hypothesis, $[(E \cup F \cup \bar{E} \cup \bar{F} \,\square\, B_i)] \subseteq T\uparrow(n-1)$ for all $1 \leqslant i \leqslant m$. Choose a ground satisfier α of $E \cup \bar{E}$. There must exist an extension β of α such that $(E \cup F \cup \bar{E} \cup \bar{F})\beta$ is true. This is because $(E \cup F \cup \bar{E} \cup \bar{F})$ is a subsystem of that obtained from the successful derivation of the original goal. Now $B_i\beta \in [(E \cup F \cup \bar{E} \cup \bar{F} \,\square\, B_i)] \subseteq T\uparrow(n-1)$ for all $1 \leqslant i \leqslant m$, and $(F \cup \bar{F})\beta$ is true. Hence, by using the clause above and the definition of T, $p(\tilde{x})\beta = p(\tilde{x})\alpha \in T\uparrow n$.

It thus remains to prove that if an atom $p(\tilde{t}) \in T\uparrow n$, $n \geqslant 0$, then $p(\tilde{t}) \in [(E \cup \bar{E} \,\square\, p(\tilde{x}))]$ where $(E \cup \bar{E} \,\square\, p(\tilde{x})) \in \mathrm{SS}$. Once again, we proceed by induction on n and the base case $n=0$ is trivial. For the induction step, let $p(\tilde{t}) \in T\uparrow n$. Thus there exists a clause $p(\tilde{x}) \leftarrow (F \cup \bar{F} \,\Box\, B_1, B_2, \ldots, B_m)$ in P and a ground substitution α such that $p(\tilde{t}) = p(\tilde{x})\alpha$, $(F \cup \bar{F})\alpha$ is true and $B_i\alpha \in T\uparrow(n-1)$ for all $1 \leq i \leq m$. By the induction hypothesis, for all $1 \leq i \leq m$, $B_i\alpha \in [(E_i \cup \bar{E}_i \,\Box\, B_i)]$ where the goal $G_i = (E_i \cup \bar{E}_i \,\Box\, B_i)$ is in SS. Let β_i be any extension of α such that $(E_i \cup \bar{E}_i)\beta_i$ is true. Since we can choose $E_i \cup \bar{E}_i$ and $E_j \cup \bar{E}_j$, $1 \leq i \neq j \leq m$, to have no variables in common, $E_1 \cup \cdots \cup E_m \cup F \cup \bar{E}_1 \cup \cdots \cup \bar{E}_m \cup \bar{F}$ is solvable; let α' be an extension of α such that α' is a ground satisfier of this system. Intuitively, α' is the union of the β_i. Using Lemma 2.3(b),

$$(E_1 \cup \cdots \cup E_m \cup \bar{E}_1 \cup \cdots \cup \bar{E}_m \,\Box\, B_1, \ldots, B_m)$$

has a successful P-derivation. Since α' solves $F \cup \bar{F}$ and since $G_i \in \mathrm{SS}$, $1 \leq i \leq m$, it easily follows that

$$(E_1 \cup \cdots \cup E_m \cup F \cup \bar{E}_1 \cup \cdots \cup \bar{E}_m \cup \bar{F} \,\Box\, p(\tilde{x})) \in \mathrm{SS}.$$

The proof is complete by noting the fact that

$$p(\tilde{t}) = p(\tilde{x})\alpha' \in [(E_1 \cup \cdots \cup E_m \cup F \cup \bar{E}_1 \cup \cdots \cup \bar{E}_m \cup \bar{F} \,\Box\, p(\tilde{x}))]. \quad \Box$$

5. Finitely failed derivations

This section is complementary to the above section in the sense that we prove $\overline{T_P\downarrow\omega} = [\mathrm{FF}(P)]$. As mentioned and exemplified above, T_P does not enjoy a closure property with respect to the usual metric space of infinite trees. It is here that it becomes apparent that Lemmas 3.2, 3.3, 3.5 and 3.6 are crucial.

Theorem 5.1. $\overline{T_P\downarrow\omega} = [\mathrm{FF}(P)]$.

Proof. Firstly we prove that $[\mathrm{FF}] \subseteq \overline{T\downarrow\omega}$. Let $G = (E \cup \bar{E} \,\Box\, p(\tilde{x})) \in \mathrm{FF}_n$, $n \geq 0$. Proceeding by induction on n, we find that the base case $n=0$ is trivial.

For the induction step, suppose, to obtain a contradiction, that $p(\tilde{x})\alpha \in [G]$ and $p(\tilde{x})\alpha \in T\downarrow n$. Using Lemma 3.3, there exists a rational substitution β such that $p(\tilde{x})\alpha \in [p(\tilde{x})\beta] \subseteq T\downarrow n$. Since $(E \cup \bar{E})$ is solvable, let β' be a rational satisfier of $E \cup \bar{E}$ above α (Lemma 3.1). Obtain the most general instance γ of β and β'. Clearly γ is rational. Now γ has both the properties $p(\tilde{x})\alpha \in [p(\tilde{x})\gamma] \subseteq T\downarrow n$ and $(E \cup \bar{E})\gamma$ is true. Let γ' be a ground and rational instance of γ. So $p(\tilde{x})\gamma' \in T\downarrow n \cap \tau^{\mathrm{R}}(\Sigma)$. By the definition of T, there must exist a ground substitution δ and clause

$$p(\tilde{x}) \leftarrow (F \cup \bar{F} \,\Box\, B_1, \ldots, B_m)$$

such that

(a) $p(\tilde{x})\gamma' = p(\tilde{x})\delta$,
(b) δ is a ground satisfier of $F \cup \bar{F}$, and
(c) $B_i\delta \in T\downarrow(n-1)$ for all $1 \leq i \leq m$.

Using Lemmas 2.1, 2.3, and 3.6, clearly, we can choose δ to be rational.

Since $(E \cup \bar{E})$ and $(F \cup \bar{F})$ have only $\tilde{x}$ as common variables and γ' and δ agree on these variables, $E \cup \bar{E} \cup F \cup \bar{F}$ is solvable. In fact, we can define a rational extension ϕ of γ' and δ such that $(E \cup F \cup \bar{E} \cup \bar{F})\phi$ is true. Using the clause above, $(E \cup F \cup \bar{E} \cup \bar{F} \,\square\, B_1, \ldots, B_m)$ is a next goal in a P-derivation of G. All P-derivations of this goal are finitely failed with length less than n. The same holds, by Lemma 2.3(c), for all goals of the form $(E \cup F \cup H \cup \bar{E} \cup \bar{F} \,\square\, B_1, \ldots, B_m)$ where H is any system of equations such that $E \cup F \cup H \cup \bar{E} \cup \bar{F}$ is solvable. Choose H to be equivalent to ϕ, i.e. ϕ is the mgu of H (Lemma 2.1(b)). Using the fact that ϕ grounds $B_1, B_2, \ldots, B_m$ (since ϕ is an extension of δ) and Lemma 2.3(d), we have that all P-derivations of $(E \cup F \cup G \cup \bar{E} \cup \bar{F} \,\square\, B_i)$, for some $1 \leqslant i \leqslant m$, are finitely failed with length less than n. Thus

$$(E \cup F \cup G \cup \bar{E} \cup \bar{F} \,\square\, B_i) \in \mathrm{FF}_{n-1}$$

and by the induction hypothesis,

$$B_i\phi \in [(E \cup F \cup G \cup \bar{E} \cup \bar{F} \,\square\, B_i)] \subseteq \overline{T{\downarrow}n-1}.$$

The desired contradiction can now be obtained from $B_i\phi = \underline{B_i\delta \in} T{\downarrow}(n-1)$.

We $\underline{\text{now}}$ prove the remaining part of the theorem, i.e. $\underline{T{\downarrow}\omega} \subseteq [\mathrm{FF}]$. Suppose $p(\tilde{t}) \in \overline{T{\downarrow}n}$, $n \geqslant 1$. By the Lemma 3.6, we have $p(\tilde{t}) \in [p(\tilde{s})] \subseteq \overline{T{\downarrow}n}$ where $p(\tilde{s})$ is a rational atom. Suppose, to obtain a contradiction, a goal G of the form $(E \,\square\, p(\tilde{x}))$, where E is equivalent to $\tilde{x} = \tilde{s}$ (Lemma 2.1(b)), is not an element of FF. Then consider the two cases

(a) G has a successful P-derivation, and

(b) G has an infinite P-derivation.

It is easy to see that (a) is impossible: suppose that a successful P-derivation of G gives a terminal goal $(F \cup \bar{F} \,\square\, \emptyset)$. Thus $(F \cup \bar{F} \,\square\, p(\tilde{x})) \in \mathrm{SS}$ and so there exists a substitution α solving $(F \cup \bar{F})$. Using the above theorem, $p(\tilde{x})\alpha \in T{\uparrow}\omega$. This is clearly a contradiction since $p(\tilde{x})\alpha \in [p(\tilde{s})]$ and $T{\uparrow}\omega \subseteq T{\downarrow}\omega \subseteq T{\downarrow}n$.

We now show that (b) is also impossible. Let the infinite P-derivation of G be given by $G_i = (E_i \cup \bar{E}_i \,\square\, \tilde{B}_i)$, $i \geqslant 0$; thus $G = G_0$. Let α be a ground satisfier of $E_n \cup \bar{E}_n$. Let I_i be the set

$$\{B\alpha : B \text{ is an atom appearing in one of } G_0, G_2, \ldots, G_i\}$$

for $i \leqslant n$. Recall that in our program clauses, all variables appearing in the atoms of the clause also appear in the system of the clause. Since $(E_i \cup \bar{E}_i) \subseteq (E_n \cup \bar{E}_n)$, $1 \leqslant i \leqslant n$, I_i contains only ground atoms. The property we desire of these sets I_i is the following: $I_{i-1} \subseteq T(I_i)$. This is easily argued by using the clauses chosen in the P-derivation and α. Now, $I_n \subseteq T{\downarrow}0$ trivially and by the monotonicity of T, $I_{n-1} \subseteq T(I_n) \subseteq T{\downarrow}1$. Extending this argument,

$$I_{n-2} \subseteq T(I_{n-1}) \subseteq T^2(I_n) \subseteq T{\downarrow}2,$$
$$I_{n-3} \subseteq T(I_{n-2}) \subseteq T^3(I_n) \subseteq T{\downarrow}3,$$
$$\vdots$$
$$I_0 \subseteq T(I_1) \subseteq T^n(I_n) \subseteq T{\downarrow}n.$$

Since $p(\tilde{x})\alpha$ is in I_0, we obtain the contradiction because $p(\tilde{x})\alpha$ is an instance of $p(\tilde{s})$. □

6. Infinite derivations

We now consider the complement II of SS $\cup$ FF, that is, $G \in$ II iff G has no successful and an infinite P-derivation. We can meaningfully partition [II] into two parts. Let α is the smallest ordinal such that $T\downarrow\alpha$ equals the greatest fixpoint of T. Such α is called the *closure ordinal* of T. We then consider the subset [IS] = [II] $\cap$ $T\downarrow\alpha$ and its complement [IF] in [II]. Diagramatically, we have Fig. 2.

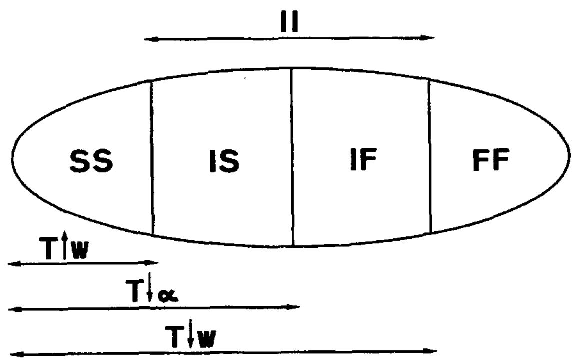

Fig. 2.

Analogous to the traditional definition (cf. [11]), we can define *ground P-derivation* as follows. A *ground goal* is of the form $(B_1, B_2, \ldots, B_n)$ where $n \geq 0$ and each $B_i \in \mathrm{HB}^*$. A ground derivation is then defined to be a (finite or infinite) sequence $G_0, G_1, \ldots$ of ground goals where for each $G_i = (B_1, B_2, \ldots, B_n)$, there exists ground substitution α and n clauses in P of the form

$$C_i \leftarrow (E_i \cup \bar{E}_i \,\Box\, C_{i1}, \ldots, C_{im_i})$$

where each of $(E_i \cup \{B_i = C_i\} \dot{\cup} \bar{E}_i)\alpha$ is true and

$$G_{i+1} = (C_{11}\alpha, \ldots, C_{1m_1}, \ldots, C_{nm_n}).$$

Where G is a (not necessarily ground) goal, a ground P-derivation of $G = (E \cup \bar{E} \,\Box\, \tilde{B})$ means a ground P-derivation of $G\alpha$ where $(E \cup \bar{E})\alpha$ is true and $\tilde{B}\alpha$ is ground.

We may now define IF as the subset of II such that $G \in$ IF if all ground P-derivations of G are finitely failed. As in traditional logic programming [11, Proposition 12], one can show the following lemma, where α is the closure ordinal of P.

Lemma 6.1. $G \in$ IF $\cup$ FF *iff* $[G] \subseteq \overline{T\downarrow\alpha}$.

The ground derivations of goals in IF have the following property which distinguishes them from goals in FF: there is no number n such that all ground P-derivations of $G \in$ II are of length $\leq n$. The following characterizes IS and IF in terms of the solvability of systems.

Lemma 6.2. (a) *The system S_ω associated with one P-derivation of $G \in \mathrm{IS}$ is solvable.*
(b) *The system S_ω associated with any P-derivation of $G \in \mathrm{IF}$ is not solvable.*

Proof. Let $G \in \mathrm{IF}$ and assume, to obtain a contradiction, that the system S_ω, corresponding to some infinite P-derivation of G, say $G_0, G_1, \ldots$, is solvable. Let α be a ground satisfier of S_ω. Thus $S_i\alpha$ is true for all $i \geq 0$, and so $G_0\alpha$, $G_1\alpha, \ldots$, is an infinite ground P-derivation of G. The contradiction then follows from the definition of IF. □

We now consider *completed programs* P^* corresponding to programs P. We say that, for the collection of all program clauses in P in the form

$$p(\tilde{x}) \leftarrow (E_1 \cup \bar{E}_1 \,\square\, B_{11}, \ldots, B_{1n_1})$$

$$p(\tilde{x}) \leftarrow (E_2 \cup \bar{E}_2 \,\square\, B_{21}, \ldots, B_{2n_2})$$

$$\vdots$$

$$p(\tilde{x}) \leftarrow (E_m \cup \bar{E}_m \,\square\, B_{m1}, \ldots, B_{mn_n}),$$

that

$$p(\tilde{x}) \leftrightarrow \left\{\begin{array}{l} (\tilde{\exists} E_1 \,\&\, \bar{E}_1 \,\&\, B_{11} \,\&\cdots\&\, B_{1n_1}) \\ \vee (\tilde{\exists} E_2 \,\&\, \bar{E}_2 \,\&\, B_{21} \,\&\cdots\&\, B_{2n_2}) \\ \quad\cdots \\ \vee (\tilde{\exists} E_m \,\&\, \bar{E}_m \,\&\, B_{m1} \,\&\cdots\&\, B_{mn_m}) \end{array}\right\},$$

where $\tilde{\exists}$ denotes existential closure, is the *completed definition* of the predicate p in P. In case a predicate symbol p does not appear in the head of a clause in P, the completed definition of p is simply

$$\tilde{\forall} \neg p(\tilde{x}).$$

Finally, the completed definition P^* of the program P is the conjunction of the completed definitions of the distinct predicate symbols in P.

We now finish off this section by establishing a strong relationship between P^* and the sets SS and FF. It is important to note that symbols such T, SS, FF, etc., which we use below are with respect to a logic program P and not to its completion P^*.

We now require a preliminary result about P^* whose proof is easily obtainable from [10].

Lemma 6.3. *I models P^* iff $T(I) = I$.*

Let A be a ground atom. The proof of the following theorem is also easily obtainable from [10].

Theorem 6.4. $P^* \models A$ *iff* $A \in [\mathrm{SS}]$.

We now address the issue of the complementary result to the above: $P^* \models \neg A$ iff $A \in [\mathrm{FF}]$. In the literature, where $\models$ stands for logical consequence, this result does indeed hold [10]. When $\models$ is interpreted to be consequence in a given structure, as is the case here, this result no longer holds. It does hold, however, for a certain class of programs.

Let us say that a program P is *derivation compact* if for each infinite sequence S_i of systems in any P-derivation it holds that if S_i, for all finite i, is solvable, then so is S_ω. Thus, for example, programs not containing unequations are derivation compact. This is easily seen by noting that any infinite set of equations is solvable iff each finite subset is solvable. Programs with unequations are, in general, not derivation compact. Take the program shown in Section 2. This program is not derivation compact because the P-derivation

$$(\square\, q(x))$$
$$(x = f(x_1),\, x \neq f(x)\, \square\, q(x_1))$$
$$(x = f(x_1),\, x \neq f(x),\, x_1 = f(x_2),\, x_1 \neq f(x_1)\, \square\, q(x_2))$$
$$\vdots$$

gives rise to an unsolvable S_ω, but each system S_i occurring at a finite stage in the P-derivation is solvable. Clearly, in general, P is derivation compact iff $\mathrm{IF} = \emptyset$.

Theorem 6.5 (Negation-as-failure). *The following statements are equivalent*:
(a) *P is derivation compact*;
(b) $P^* \models \neg A$ *iff* $A \in [\mathrm{FF}]$.

Proof. In one direction, i.e. (b) implies (a), we have that (b) implies that the greatest model of P^* is $\overline{[\mathrm{FF}]} = T{\downarrow}\omega$. This in turn implies, by Lemma 6.3 that $T{\downarrow}\omega = \mathrm{gfp}(T)$. Thus $\mathrm{IF} = \emptyset$ and we are done.

Suppose now that (a) holds and we show that (b) follows. If $A \in [\mathrm{FF}]$, then $P^* \models \neg A$ by Lemma 6.3. For the converse result, let $A \in \overline{[\mathrm{FF}]}$ and we may as well assume that $A \in \overline{[\mathrm{SS}]}$ because of the theorem above. Using Lemma 3.6 and the results of Section 5, $A \in [G] \subseteq \overline{[\mathrm{FF}]}$ for some goal G. As in the proof Lemma 6.2, we have an infinite ground P-derivation of G, say $G_0\alpha$, $G_1\alpha$, ... We now define the set

$$I = \{B\alpha : B \text{ is an atom appearing in } G_i,\ i \geq 0\}.$$

The property we desire of I is $I \subseteq T(I)$ and this is easily proven as follows: let $C \in I$. Thus $C = B\alpha$ for some B appearing in G_n for some $n \geq 0$. Letting

$$I' = \{B\alpha : B \text{ is an atom appearing in } G_{n+1}\},$$

it is now an easy matter to verify that $C \in T(I')$ and thus $C \in T(I)$ by the monotonicity of T. Finally, we use a well-known fixpoint theorem that there exists a J containing I such that $T(J) = J$. Now $A \in [G] \subseteq J$ and by using Lemma 6.3 yet again, we are done. $\square$

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